The wind in your sails.

An introduction to right-angled trigonometry through the art of sailing.

Written by Alastair Lupton and Anthony Harradine

TAKE THE HELM >>
The wind in your sails.


Written by Anthony Harradine and Alastair Lupton

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Using this resource.

This resource is not a textbook.

It contains material that is hoped will be covered as a dialogue between students and teacher and/or students and students.

You, as a teacher, must plan carefully ‘your performance’. The inclusion of all the ‘stuff’ is to support:

• you (the teacher) in how to plan your performance – what questions to ask, when and so on,
• the student that may be absent,
• parents or tutors who may be unfamiliar with the way in which this approach unfolds.

Professional development sessions in how to deliver this approach are available.

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Legend.

EAT - Explore And Think.
These provide an opportunity for an insight into an activity from which mathematics will emerge – but don’t pre-empt it, just explore and think!

At certain points the learning process should have generated some burning mathematical questions that should be discussed and pondered, and then answered as you learn more!

Time to Formalise.
These notes document the learning that has occurred up to this point, using a degree of formal mathematical language and notation.

Examples.
Illustrations of the mathematics at hand, used to answer questions.
1. **Take the helm!**

Have you ever sailed in a yacht? Do you know how to sail? If not, don’t worry as you are able to gain some ideas about sailing from a simulator. A simple one can be found at:


1.1 Task 1.

Using the simulator, adjust the rudder and sail controls until you have the yacht sailing as fast as possible. **Record your top boat speed and the wind speed and summarise your sailing strategy.**

1.2 Task 2.

Adjust the rudder controls until you have the yacht sailing due east. Now adjust your sail control to maximise your speed. **Record your top boat speed, the wind speed, the wind direction and summarise your sailing strategy.**

1.3 Task 3.

Adjust the rudder controls until you have the yacht headed directly into the wind. Now adjust your sails to get your yacht travelling at 5 knots or more. If you cannot do this, head your boat slightly out of the wind and adjust the sails so that you can travel at 5 knots. **How ‘close to the wind’ can you sail and still achieve a speed of 5 knots?**
2. **A geometric model of sailing on a 30 degree tack.**

To sail a boat, using wind as the only source of power, is a skilful activity. With the right technique, it is possible to sail in almost any direction, regardless of where the wind is blowing from.

The only limitation is that a boat cannot sail directly into the wind, or too close to that direction.\(^1\)

This can mean that, for example, if you want to sail east and the wind is blowing from the east, you are forced to sail slightly off course.

This is referred to as sailing on a **tack**. The result of a tack is that a boat ends up **off course**, by a distance that has to be compensated for when the boat changes direction.

The calculation of the distances involved in a tack is important for sailors.

For example, if a tack of \(30^\circ\) (away from the wind direction) is sailed for given distance, how far off course will the boat be?

### 2.1 EAT1

a. Generate a random integer between 1 and 30 (you may wish to use your graphics calculator). This will be the distance of your \(30^\circ\) tack, in kilometres.

b. Using a ruler and geolinier, draw a **scale drawing** of your tack on a piece of graph paper. Measure your distance off course.

c. Add 2 or 3 more tack lengths (and distances off course) to your diagram – generate other random integers or use the values generated by your neighbours.

**Pool each class member’s result, displaying them in a number of ways.**

**Will You Look At That! (WYLAT!)**

---

\(^1\) The degree that a boat can sail into the wind depends on the strength of the wind and the characteristics of the boat.
Following EAT 1 it is hoped you have had a few questions pop into your head due to your natural mathematical inquisitiveness. List the questions and share them with your classmates.

2.2 EAT 2

On your Classpad 300, open the geometry file called atackman. This is a geometric model of a boat sailing. Select the diagonal line representing Dist. Sailed. Use the measurement menu bar (tap [2]) to enter four values of your choosing for OB.

Pool each class member’s results, displaying them in a number of ways.

Will You Look At That! (WYLAT!)

As a class we now have numerous accurately measured cases of this interesting phenomenon. Does this raise any further questions?

2.3 EAT 3

Open the geometry file called btrackani. In this file you can sail the boat manually. Just tap point B and move it to a new location.

Or, you can see the boat in ‘full sail’ via a simple animation, accessed via Edit : Animate : Go(once).

While sailing, the ClassPad 300 has been collecting data. Make a table of measurements based on this animation that may help to answer the questions you have.

Paste this data into a ClassPad 300 spreadsheet. Use the data to see if you can gain support for your thoughts.
By now you should be fairly confident that:

**For a tack of 30° of any length the ratio of the distance Off Course to Dist. Sailed is constant and has the value 0.5.**

We can think about this in a number of ways:

- \( \frac{\text{Off Course}}{\text{Dist. Sailed}} = \frac{1}{2} \)
- As a set of ‘nested’ similar triangles
- \( \text{Dist. Sailed} = 2 \times \text{Off Course} \)
- For every 2 unit of distance you sail on a tack of 30°, you will be 1 unit of distance off course.

Clearly, this is a useful piece of information for sailors who are on a tack of 30°, but ……………………….
3. A geometric model of sailing on a tack of degrees other than 30.

In the previous section we gained confidence in a very specific result. It is hoped that you have asked some questions as a result that lead to the investigation of whether the result can be generalised, i.e. does it hold for angles other than 30°? Two key questions are:

- Is \( \frac{\text{Off Course}}{\text{Dist. Sailed}} \) constant for other angle values?
- If it is, what is the ratio for other angle values?

3.1 EAT4

Investigate the ratio \( \frac{\text{Off Course}}{\text{Dist. Sailed}} \) for three other angles of your choosing. Be sure to accurately measure many cases before making a conjecture.

Pool your findings with your classmates and display your collective findings in a number of ways.
4. Formalising our findings 1 – the sine ratio.

As a result of these investigations you should be confident that the ratio \( \frac{\text{Off Course}}{\text{Dist. Sailed}} \) is constant (but different) for all angles.

In other words the fraction \( \frac{\text{Off Course}}{\text{Dist. Sailed}} \) has the same value, regardless of the size of the tack, and this value changes depending on the angle of the tack.

This relationship was discovered and proven to be true many years ago. It is an important aspect of Trigonometry, the mathematical study of triangles. This relationship has very wide applications.

Our tacks can be thought of as right angled triangles, because the intended direction and the distance off course are always perpendicular.

Right-Angled Trigonometry focuses on the link between one of the angles in a right angled triangle (not the 90° angle) and the ratio of pairs of triangle side lengths.

In Trigonometry the following is commonly used language:

- \( \theta \) – the angle being studied/known,
- hypotenuse (hyp.) – the longest side, the side opposite the right angle,
- opposite side (opp.) – the side opposite the angle being studied/known \( \theta \)
- adjacent side (adj.) – the side forming the angle \( \theta \) (but not the hypotenuse)

We are now confident that, for a tack of 30°, \( \frac{\text{Off Course}}{\text{Dist. Sailed}} = \frac{1}{2} \).

This ratio is called as the sine of 30° (i.e. \( \sin 30^\circ = \frac{1}{2} \)).

More generally we say:

\[
\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}
\]

(sine can be written sin but is always pronounced ‘sign’).
Traditionally, sine values for any angle (to a set degree of accuracy) were published in table form and used by sailors, amongst others. This is an excerpt from one such a table,

<table>
<thead>
<tr>
<th>θ</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin θ</td>
<td>0</td>
<td>0.174</td>
<td>0.342</td>
<td>0.5</td>
<td>0.643</td>
<td>0.766</td>
<td>0.866</td>
<td>0.940</td>
<td>0.985</td>
<td>1</td>
</tr>
</tbody>
</table>

While there may appear to be no particular pattern to the sine values, drawing a graph of sine of θ vs. θ reveals a clear pattern/trend.

Though discussion with your classmates, decide upon an agreed description of how the value of sin θ changes as the value of θ changes.
The sine ratio can be used to compute unknown distances. We must know the value of one angle in a right-angled triangle and the length of either the opposite side or the hypotenuse. Two examples of this are given for you to consider.

Example 1

Suppose a boat sailed on a tack of $40^\circ$ to its intended direction for 15 km. How far will it have sailed off course?

Using the relationship $\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$ in the triangle, and the fact that $\sin 40^\circ = 0.643$ we can say:

$$0.643 = \frac{\text{off course}}{15 \text{ km}} \quad \text{and so} \quad \text{off course} = 0.643 \times 15 \text{ km}$$

Therefore the boat will be $9.645 \text{ km}$ off course.

Example 2

If a boat on a tack of $20^\circ$ is off course by 72 km how far has it sailed (to the nearest km)?

As $\theta = 20^\circ$, the opposite side is $72 \text{ km}$ long, and if we let $x$ represent the unknown distance sailed, then we have:

$$\sin 20^\circ = \frac{72}{x}\quad \Rightarrow \quad 0.342 = \frac{72}{x}\quad \Rightarrow \quad x \times 0.342 = 72\quad \Rightarrow \quad x = \frac{72}{0.342}$$

Hence, the boat will have sailed 211 km.
4.1 Can you use the knowledge? 1

Use the table on page 11 to determine how far off course a boat is if it sails on
1. a tack of 20° to the intended direction for a distance of 4 km?
2. a tack of 70° to the intended direction for a distance of 1.88 km?
3. a tack of 50° to the intended direction for a distance of 400 m?

Use the table on page 11 to determine how far a boat has sailed if it on
4. a tack of 80° to the intended direction and is 73.9 km off course?
5. a tack of 40° to the intended direction and is 386 m off course?
6. a tack of 60° to the intended direction and is 32.2 km off course?
5. Automating our new-found knowledge 1 – the sine ratio and unknown sides.

So that you are able work with the sine ratio efficiently and so that you can use it in broader uses in later work, it is important that you automate the process of using the sine ratio. To help you reach a high level of automation, consider the following input and examples and then complete the ‘automation sets’.

As we have seen, $\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$

This is a ‘formula’ into which values can be substituted and for which unknowns can be calculated.

To compute the sine ratio for a given angle, rather than use the tables (which is a tad ‘empty’) we can use any scientific calculator as the ‘full’ table is available on them.

To compute $\sin 45^\circ$ on the ClassPad 300 go to $\text{Ans}$ and, using the $\text{TRIG}$ keyboard, enter $\sin(45)$.

If your calculator returns a different value then your calculator is set to expect angles to be entered in Radians rather than degrees. (Radians is the default setting).

If your calculator returns a value in surd form then it is set to Standard not Decimal calculation mode.

- To change the angle setting on a ClassPad 300, tap on the word Rad on the bottom of your screen and it will change to Deg.

- To change the calculation mode on a ClassPad 300, tap on the word Standard on the bottom of your screen and it will change to Decimal.
Unknown sides of right-angled triangles can be found as follows:

**Example 3**

\[
\sin 36° = \frac{x}{13.2 \text{ cm}}
\]

\[
\therefore x = \sin 36° \times 13.2 \text{ cm}
\]

\[
\therefore x = 7.76 \text{ cm}
\]

**Example 4**

\[
\sin 27° = \frac{0.29 \text{ km}}{y}
\]

\[
\therefore y \times \sin 27° = 0.29 \text{ km}
\]

\[
\therefore y = \frac{0.29 \text{ km}}{\sin 27°}
\]

\[
= 0.639 \text{ km}
\]

**Note:** The value of the sine ratio for the angle does not necessarily need to be written down. It can be ‘called up’ in the calculator in the last step when the answer is evaluated. In this way 15 significant figure accuracy is used by the calculator (with around 10 figures displayed).

Rounding of the answers to 3 significant figures was used in the above examples.

### 5.1 Automation Set 1

Find the unknown side lengths correct to 3 significant figures.

1. \[11.9 \text{ m} \quad 23° \quad x\]
2. \[y \quad 0.3 \text{ mm} \quad 166 \text{ km} \quad 53°\]
3. \[18° \quad z\]
4. \[a \quad 78 \text{ cm} \quad 34°\]
5. \[22° \quad 14.1 \text{ m} \quad b\]
6. \[1.34 \text{ km} \quad c \quad 68°\]
6. Automating our new-found knowledge 2 – the sine ratio and unknown angles.

The sine relationship \( \sin \theta = \frac{\text{opp}}{\text{hyp}} \) contains three values \( \text{opp} \), \( \text{hyp} \) and \( \theta \).

If any two of these values are known the third one can be found.

We have found the lengths of the opp (and hyp) already, when \( \theta \) and the ‘other’ side were known. So if the opp side and the hyp were known, we should be able to find the angle \( \theta \). But how?

**Example 5**

Find the size of the angled marked as \( \theta \).

\[
\sin \theta = \frac{10.7}{18.4} \quad \Rightarrow \quad \theta = \text{the angle with a sin of } \frac{10.7}{18.4}
\]

\[
\Rightarrow \quad \theta = ??? \quad (\text{Read on below}).
\]

“The angle with a sine of \( \frac{10.7}{18.4} \)” can be found by computing the ‘inverse’ sine of the ratio \( \frac{10.7}{18.4} \).

This is more formally written as \( \sin^{-1} \left( \frac{10.7}{18.4} \right) \), which can be computed using your calculator, as seen below (continuing from above).

\[
\sin \theta = \frac{10.7}{18.4}
\]

\[
\therefore \quad \theta = \sin^{-1} \left( \frac{10.7}{18.4} \right) \quad (\sin^{-1} \text{ is on the TRIG keyboard})
\]

\[
\therefore \quad \theta = 35.6^\circ \quad \text{correct to 3 sig. fig.}
\]

**Note that, the inverse sine is also called the arcsin.**
6.1 Automation Set 2

1. Find the value of $\theta$, the unknown angle, to the nearest tenth of a degree.
   a. 
   b. 
   c. 

2. Find the value of the unknowns to three significant figures.
   a. 
   b. 
   c. 
   d. 
   e. 
   f. 

3. The following is entered in a ships log:

   - Tack 1: 15.5 degrees for 4.8 km – 1.28 km off course
   - Tack 2: 37.2 degrees for 1.7 km – 1.28 km off course
   - Tack 3: 20.5 degrees for 16.4 km – 6.93 km off course

   The captain thinks errors have been made – is the captain correct?

4. The manual of an extension ladder says
   for safe operation this ladder must not be used at an angle of greater than 70° to the ground
   a. Will a 3.2 m ladder safely reach a point 3 m up a wall?
   b. How long a ladder is needed to reach a point 5.5 m up a wall?
   c. Is a 2.4 m ladder reaching 2.1 m up a wall being safely operated?
   d. What assumption is necessary for these calculations to be made?
7. **Tacking – an efficiency rating.**

Another way to quantify tacking is to focus on the distance travelled in the intended direction (the Distance Intended) of the boat during a tack.

Information about this is also of use in boat navigation.

### 7.1 EAT5

Launch the geometry application on your graphics calculator and open the geometry file `ctacmeff`. For a tack of 30° select the line representing Dist. Sailed. Use the measurement menu bar (tap `u`) to enter four values of your choosing for the length OB. For each distance sailed, record the appropriate information. **Pool your findings with your classmates and display your collective findings in a number of ways.**

What question(s) does this EAT activity raise?

### 7.2 EAT6

Use geometry file (`dtacaeff`) and/or other methods to investigate the relationship between Dist. Sailed and Distance Intended for a variety of different tack angles (including 60° amongst others). **Pool your findings with your classmates and display your collective findings in a number of ways.**
8. **Formalising our findings 2 – the cosine ratio.**

As a result of the previous EATs, you should have seen that the ratio \( \frac{\text{distance intended}}{\text{distance sailed}} \) is constant for a given tack angle, and for each angle this ratio takes a different value.

You may also have noticed that this ratio works like an efficiency rating,

- Starting at 1 - if you were to sail on a tack of \( 0^\circ \)
  
  (100% in your intended direction – complete efficiency)

- Ranging down to 0 if you were to sail on a tack of \( 90^\circ \)
  
  (perpendicular to your intended direction – no progress)

A table of sine values and our new ratio is given below.

Study this table, what do you notice?

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>0.174</td>
<td>0.342</td>
<td>0.5</td>
<td>0.643</td>
<td>0.766</td>
<td>0.866</td>
<td>0.940</td>
<td>0.985</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\text{distance intended}}{\text{distance sailed}} )</td>
<td>1</td>
<td>0.985</td>
<td>0.940</td>
<td>0.866</td>
<td>0.766</td>
<td>0.643</td>
<td>0.5</td>
<td>0.342</td>
<td>0.174</td>
<td>0</td>
</tr>
</tbody>
</table>

You should see that the ratio of \( \frac{\text{distance intended}}{\text{distance sailed}} \) for a given angle \( \theta \) is the same the sine ratio \( \frac{\text{off course}}{\text{dist.sailed}} \) for the complementary angle \( 90 - \theta \). It is for this reason that the ratio of \( \frac{\text{distance intended}}{\text{distance sailed}} \) is known as the complement of sine or cosine for short.

A graph of \( \cos \theta \) vs \( \theta \) can be seen below (solid curve).
We say that:

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

(cosine can be written as the abbreviation ‘cos’)

The cosine ratio can be used in a similar way as the sine ratio to find the length of unknown sides or angles in a right-angled triangle, given we know some other information.

When the cosine of an angle is required, just call up the required value using the calculator.

Consider the following examples, the first where an unknown side is found and the second where an unknown angle is found.

**Example 6**
Find the length of the side defined as $x$.

$$\cos 54^\circ = \frac{11.4}{x}$$

$$\Rightarrow x \times \cos 54^\circ = 11.4$$

$$\Rightarrow x = \frac{11.4}{\cos 54^\circ}$$

$$\Rightarrow x = 19.4 \text{ km}$$

**Example 7**
Find the size of the angle defined as $\theta$.

$$\cos \theta = \frac{23}{31}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{23}{31}\right)$$

$$\Rightarrow \theta = 42.1^\circ$$
8.1 Automation Set 3

1. Find the value of the unknowns to 3 significant figures.
   a. \( y \)
   b. \( 8 \text{ m} \)
   c. \( x \)

   \[
   \begin{align*}
   &69^\circ \quad 3.45 \text{ mm} \\
   &\theta \\
   &13 \text{ m} \\
   &27^\circ \\
   &13.1 \text{ cm}
   \end{align*}
   \]

   d. \( 111 \text{ mm} \)
   e. \( h \)
   f. \( 54^\circ \)

   \[
   \begin{align*}
   &147 \text{ mm} \\
   &\theta \\
   &350 \text{ m} \\
   &77^\circ \\
   &1.26 \text{ km}
   \end{align*}
   \]

2. Answer parts d and e without the use of sine.

3. The cosine ratio for \( 25^\circ \) is 0.906. Interpret this value in terms of a boat sailing on a tack of \( 25^\circ \).
9. Extending our knowledge – the tangent ratio.

So far we have arrived at the two following ratios:

\[
\cos \theta = \frac{\text{adj side}}{\text{hyp}} \quad \sin \theta = \frac{\text{opp}}{\text{hyp}}
\]

Look closely at the composition of the ratios. They both contain a divisor of the length of the hypotenuse. What does this suggest might be possible?

Consider the following:

\[
\frac{\sin \theta}{\cos \theta} = \frac{\frac{\text{opp}}{\text{hyp}}}{\frac{\text{adj}}{\text{hyp}}} = \frac{\text{opp}}{\text{adj}}
\]

Dividing returns a new derived ratio that is free of the hypotenuse length. This could be very useful for problems where we have no immediate knowledge of the hypotenuse.

This is called the tangent ratio, commonly shortened to tan.

It is defined as \[ \tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} \]

9.1 Can you use the knowledge? 2

Use the above definition and what you already know about sine and cosine to answer the following questions

1. What is the value of \( \tan 0 \)?
2. When will \( \tan \theta = 1 \)?
3. What will happen to \( \tan \theta \) as \( \theta \) increases?
4. Discuss the value of \( \tan 90 \).
5. Complete the table below,

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>0.174</td>
<td>0.342</td>
<td>0.5</td>
<td>0.643</td>
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<td>0.866</td>
<td>0.940</td>
<td>0.985</td>
<td>1</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>0.985</td>
<td>0.940</td>
<td>0.866</td>
<td>0.766</td>
<td>0.643</td>
<td>0.5</td>
<td>0.342</td>
<td>0.174</td>
<td>0</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A graph of all three ratios vs $\theta$ is given below.

Note that the tangent ratio for $90^\circ$ is undefined.

What problems might this third ratio help to solve?
The tangent ratio can be used whenever the hypotenuse is not involved in calculations. One common application is to determine the vertical height of objects that are hard to directly measure, but where a horizontal distance and angle of elevation can be measured.

This can be something like the tree shown.
For example, if the angle of elevation is $40^\circ$ when a person with an eye height of 1.5 metres is standing 10 metres from the base of the tree then, using the following tan calculation,

**Example 8**

\[
\tan \theta = \frac{\text{opp}}{\text{adj}}
\]

\[
\tan 40^\circ = \frac{\text{opp}}{10}
\]

$\Rightarrow \quad \text{opp} = 10 \times \tan 40^\circ$

$\Rightarrow \quad \text{opp} = 8.4\text{ metres}$

Based on this, the tree’s height is 9.9 metres (when the eye height of the observer is taken into account).
10. **The three major trigonometric ratios.**

We can now summarise the three major trigonometric ratios.

\[
\begin{align*}
\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \tan \theta &= \frac{\text{opp}}{\text{adj}}
\end{align*}
\]

10.1 **Automation Set 4**

1. Find the value of the unknown lengths/angles to 3 sig. fig’s.
   a. 
   b. 
   c. 
   d. 
   e. 
   f. 
   g. 
   h. 
   i. 
   j. 
   k. 
   l.
2. A diagonal brace joins opposite corners of a rectangular gate measuring 900 mm by 1250 mm. What angle does this brace make with the long side of the gate?

3. A boat sails 14 km east then 5 km north. It then sails straight back to its starting position. What is the distance and bearing of the return journey?

4. Find the area of the right angled triangle with a hypotenuse 50 cm that contains an angle of 42°.
11. Application tasks

11.1 Application Task 1
Using a method similar to the one discussed on page 23, determine the height of something near you, like a flag pole, tree or building. You will need measuring equipment. Compare your answer with others in the class who have ‘measured’ this same thing. Discuss any discrepancies and assumptions in your work.

In the last activity you may have experienced discrepancies between different ‘measurements’ of the same thing and might have wondered about their origin. This sort of calculation relies on two assumptions

- The right angled triangle i.e. the measured object is actually vertical and the ground is level and horizontal.
- The measurements were accurate.

It is this second point that causes the discrepancies you saw.

So how big is the effect of measurement error?

11.2 Application Task 2
As a class decide on an estimate for the greatest error in your measurement of the horizontal distance. Hence calculate a range of height values - based on the range of possible horizontal distances. Repeat this process for the angle measurement.

Which measurement seems to cause greater discrepancies?
See eTech Support for a possible approach to this activity.

Why do we care so much about the height of trees?

Forestry plantations are a valuable and carefully managed resource. To maximise the profits made the time of harvest is crucial. To make this decision calculations about the volume of wood in the plantation are needed. These calculations rest upon accurate values for the width and height of the trees in the plantation.
12. Contextual questions.

By now we hope you have automated the process of applying the three trigonometric ratios to simple triangles. It is also important to be able to solve questions posed about contexts. The following set of questions will provide you with the opportunity to develop this skill.

We suggest that you:

- Familiarise yourself with a clearly drawn diagram.
- Look for useful geometric features, including right angles.
- Identify the unknown(s) that you wish to find values for.
- Identify the known information, marking it on the diagram if possible.
- Work systematically from what you know to what you want to find out.

12.1 Contextual Set 1

1. Find the value of all unknown sides and angles in the following triangles to the nearest whole unit.

   a. \[ \begin{array}{c}
   \text{35°} \\
   \text{90°} \\
   \text{55 cm}
   \end{array} \]

   b. \[ \begin{array}{c}
   \text{92 m} \\
   \text{36 m}
   \end{array} \]

   c. \[ \begin{array}{c}
   \text{22 km} \\
   \text{56°}
   \end{array} \]

2. Use a diagram to help you answer these questions.

   a. A plane flies on a bearing of 37°T for 1750 m. How far north has it flown?

   b. A hiker walks 1.2 km south then 1.9 km west. What is the bearing of their new position from their starting point?

   c. Wishing to sail west, a boat sails on a tack of 306°T for 2.7 km before sailing on a course of 228°T for 2.4 km. How far west has the boat sailed and how far off course is it?
d. An expedition travels on a bearing of $110^\circ T$ for 45 km before heading on a bearing of $20^\circ T$ for 72 km. Find the distance and bearing of the return leg of their trip.

Does this look familiar...?  

![Any isosceles triangle can be bisected thus...](image)

3. Find the value of $x$ in these isosceles triangles to 3 sig. fig’s.
   a. 
   ![isosceles triangle](image)
   
   b. 
   ![isosceles triangle](image)
   
   c. 
   ![isosceles triangle](image)

4. An isosceles triangle has side lengths of 2, 2, and 3 cm. Find its smallest angle.

5. Find the area of an isosceles triangle with a repeated side length of 14 cm and an apex angle of $100^\circ$.

6. How high is the mountain...measure one without climbing it?

To find the height of this mountain an angle of elevation of $22^\circ$ is taken of the mountains peak. Then, from 1230 metres directly closer, and at the same altitude, another angle of elevation is taken, this time of $30^\circ$.

a. Use the first elevation angle to write down a trigonometric ratio.
b. Do similarly using the second elevation angle.

c. Re-arrange these equations into the form $h = ...$

d. Find the height of the mountain by finding the simultaneous solution of these two equations.

e. Why might this height differ from the value quoted as the ‘actual’ height of this mountain.

7. A try is scored in a game of Rugby (at point C below). A kick at goal is taken from somewhere along the line from C that perpendicular to the goal line. The goal kicker chooses the point (O) that he kicks from. A diagram of a rugby pitch is shown below,

Imagine that a try is scored 10 metres from the right hand goal post.
The margin for error in kicking can be thought of as the angle of view $\angle AOB$.
So, how does the choice of point O affect this margin for error?

a. If possible, go to a nearby sports field and make your own choice for point O.
b. Alternatively, open the geometry file RUGBYKCK and use this geometric simulation to get a ‘feel’ for the optimal position for O.
c. Find the margin for error $\angle AOB$ in the case where the kick is taken from point O when it is 5 metres from the goal line.
d. Repeat part c when the distance of O from the goal line is
   i. 10 metres    ii. 15 metres   iii. 20 metres.
e. Describe what seems to happen to the margin for error as the distance of O from the goal line increases.
f. Determine from where the goal kicker should kick the ball.
   What factors will influence this discussion? Discuss with your class.
13. Approximate or exact?

Did you notice that...

\[ \sin 30^\circ = \frac{1}{2} \] (nice and neat) but \[ \cos 30^\circ = 0.866... \] (a messy decimal)

...and vice versa for \( \sin 60^\circ \) ?

13.1 EAT7

Is there a neater value for \( \cos 30^\circ \) ?

Consider an equilateral triangle with side lengths of 1 unit.

Bisect it to create a right angled triangle (or two).

Use this construction to help you find exact values for \( \cos 30^\circ \) (and therefore also for \( \sin 60^\circ \)).

13.2 EAT8

What about \( \sin 45^\circ \) ?

Consider bisecting a square with sides of 1 unit to find an exact value for \( \sin 45^\circ \) (and therefore also for \( \cos 45^\circ \)).

Use these constructions (plus a little extra thought) to copy and complete this table of exact trigonometric values.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is possible to obtain exact trigonometric values for other angles (like \( 15^\circ \)). The table above includes all the widely used exact trigonometric values. They are worth remembering if possible as they can come in quite useful in future work.
13.3 Automation Set 5

Use these exact trigonometric values to find the exact value of x in the following shapes.

a. ![Diagram a](image)

b. ![Diagram b](image)

c. ![Diagram c](image)

d. ![Diagram d](image)

e. ![Diagram e](image)

f. ![Diagram f](image)
14. eTech Support.

14.1 Generating random numbers.
Random numbers between 0 and 0.999... are generated by the command \texttt{rand(}.
This can be accessed via the \texttt{cat} menu.
Using the \texttt{cat} menu, commands can be found alphabetically.

Multiplying this command by a constant (like 30) ‘stretches’ the range of these random numbers from 0 to 29.999... . By adding 1, values from 1 to 30.999... are obtained.

To return just the integer part of this, the command needs to be prefaced by \texttt{int(}. This command can also be found in the \texttt{cat} menu.

14.2 Opening a geometry file.

Tap \texttt{G} in the \texttt{Menu} of your ClassPad 300.
Tap \texttt{File then Open}.
Tap the black wedge to see the contents of the Sailing folder.
Tap atackman and then tap \texttt{Open}.

14.3 Selecting and altering the size of geometric features.

Tap on the line segment OB.
Access the measurement menu bar by tapping \texttt{u}.
Enter your chosen lengths and press \texttt{w}. 
14.4 Grabbing and dragging geometric features.

Tap on point B so that it is highlighted (black square).
Place your cursor on the black square and slide it to its new location.

14.5 Running a geometry animation and tabulating.
To run a previously created animation tap
Edit : Animate : Go(once)

The values used in an animation (i.e. the distances at each step along the way) can be tabulated in the following way,

Tap on the Off Course line segment BA.
Tap and then tap on the table icon #.

To add to this table, open the Memory window.
Tap on clear space to deselect the current selection.
Tap on points O and B.
Now tap on # again to add the lengths of OB to your table.

To paste this table into a ClassPad 300 spreadsheet,
whilst in the table window, tap Edit : Select All then
tap Edit : Copy.

Now go to Spreadsheet and, with cell A1 selected,
tap Edit : Paste
14.6 Working with your table in a spreadsheet.

If your conclusion was that the length Off Course is half of the length Dist. Travelled then this can be checked by dividing Column A by Column B.

This calculation can be made by tapping Edit : Fill Range and entering the inputs as shown,

14.7 Entering an error function.

(Based on a horizontal distance of 10 metres and a greatest error estimate of ± 0.2 metres)

The height calculation of \( \text{adj} \times \tan \theta \) could be evaluated for \( \text{adj} = 9.8, 9.85, \ldots, 10.15, 10.2 \).

(using a fixed angle of elevation of 40°)

in W mode by entering the height as a function

To set the table inputs tap on \( \square \) and enter,

Now tap the table icon \( \square \) to produce a table that shows a variation in calculated height of less than 40 cm due to this degree of error in horizontal distance.

A similar approach to possible error involving the angle of elevation would yield a table something like the one on the right that shows that angle errors of 4 degrees or less can cause the calculated height to vary by nearly two and a half metres!
15. Answers.

Can you ....1.

1. 1.368 km
2. 1.7672 km
3. 306.4 m
4. 75.025 km
5. 600.3 m
6. 37.18 km

Auto. Set 1

1. 4.65 m
2. 133 km
3. 0.971 mm
4. 43.6 cm
5. 37.6 m
6. 1.45 km

Auto. Set 2

1.
   a. 27.7°
   b. 18.6°
   c. 48.7°
2.
   a. 13.2 mm
   b. 53.1°
   c. 3.99 m
   d. 47.2°
   e. 6.50 m
   f. 51.3°

Auto. Set 3

1.
   a. 1.23 mm
   b. 52.0°
   c. 14.7 cm
   d. 49.0°
   e. 359 m
   f. 0.741 km
3.
   For every 1 unit of distance travelled, 0.906 units of progress is made toward your destination.

Auto. Set 4

1.
   a. 7.55 m
   b. 1580 m
   c. 39.9°
   d. 2.59 km
   e. 71.9°
   f. 0.000694
   g. 26.5°
   h. x=31.2 cm, y=13.3 cm, z=67°
   i. 522 m
   j. a=50.1°, b=39.9°, c=66.1 km

Can you ....2.

1. 0
2. when $\theta = 45°$
3. It increases, at a greater and greater rate.
4. As $\theta \to 90$, $\tan \theta$ gets very large. At 90° its exact value is undefined.
k. $36.5^\circ$

l. $2.58 \text{ cm}$

2. $35.8^\circ$

3. $14.9 \text{ km}, 70.3^\circ \text{ T}$

4. $623.1 \text{ cm}^2$

**Context Set 1.**

1.

a. $55^\circ, 39 \text{ cm} 67 \text{ cm}$

b. $21^\circ, 69^\circ, 99 \text{ m}$

c. $34^\circ, 18.2 \& 12.3 \text{ km}$

2.

a. $1398 \text{ m}$

b. $238^\circ \text{ T}$

c. west $3.96 \text{ km}$

d. off course $0.02 \text{ km}$

d. $85 \text{ km}, 232^\circ \text{ T}$

3.

a. $13.74 \text{ cm}$

b. $92.2^\circ$

c. $11.36 \text{ mm}$

4. $41.4^\circ$

5. $96.3 \text{ cm}^2$

6.

a. $\tan 22^\circ = \frac{h}{x + 1230}$

b. $\tan 30^\circ = \frac{h}{x}$

c. $h = (x + 1230) \tan 22^\circ$

d. $1655 \text{ m}$

7.

a. $8.69^\circ$

b.

i. $12.17^\circ$

ii. $12.25^\circ$

iii. $11.21^\circ$

c. It seems to increase then decrease.

**Auto. Set 5**

a. $2$

b. $20$

c. $100 \sqrt{2}$

d. $60^\circ$

e. $22.5$

f. $45^\circ$